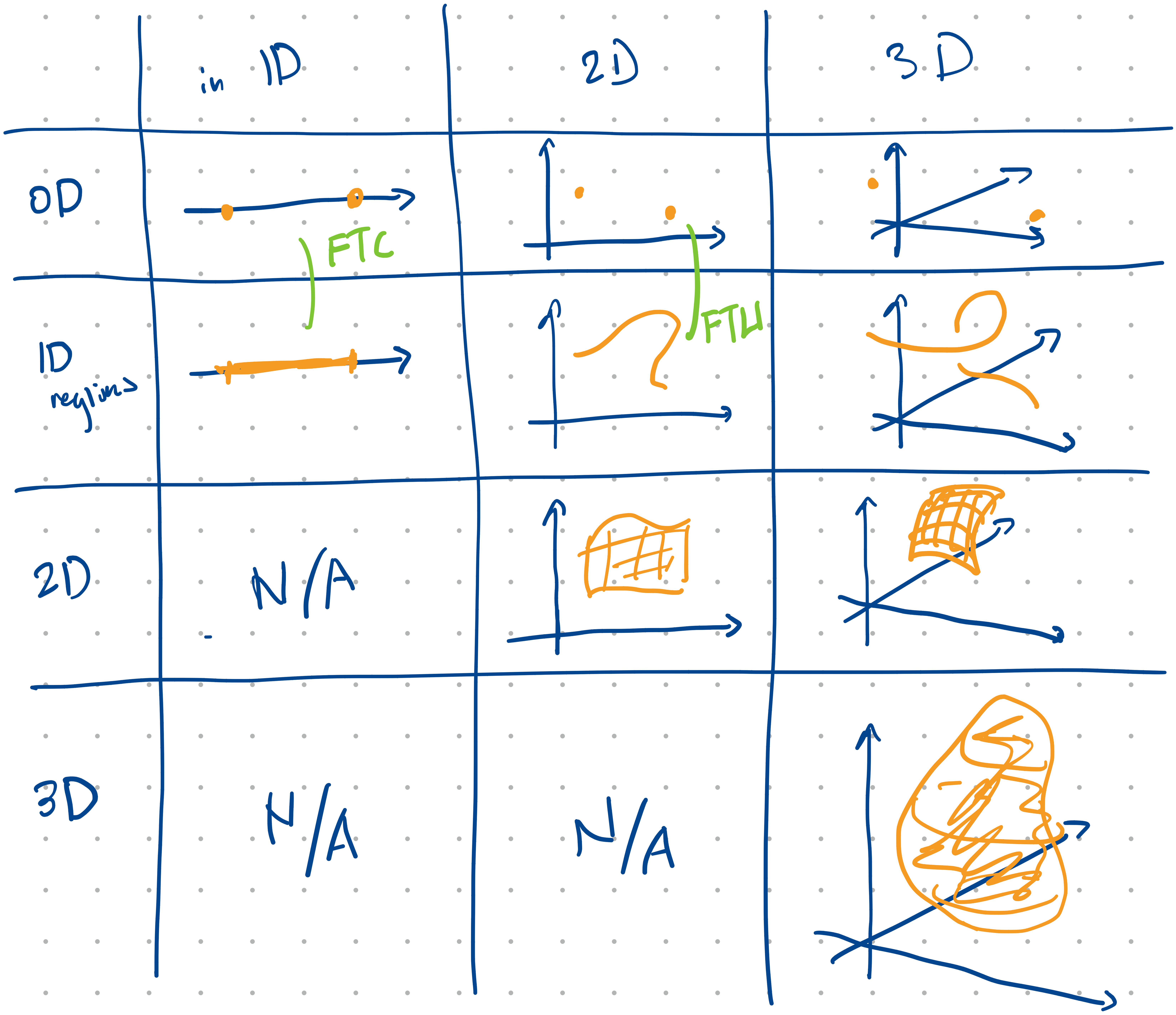


⚠ FTLI, "path-indep" etc. are about <u>vector fields</u>.

i.e. they are about

$$\int_C \vec{F} \cdot d\vec{r} \qquad \underline{\text{NOT}} \qquad \int_C f \, ds$$

SVC: scalar fn. $\xrightarrow{\text{diff.}}$ ① scalar function

e.g. $f(x) = x^2$ $\quad\quad$ $f'(x) = 2x$.

MVC: scalar fn $\xrightarrow[\nabla]{\text{diff}}$ ② vector field

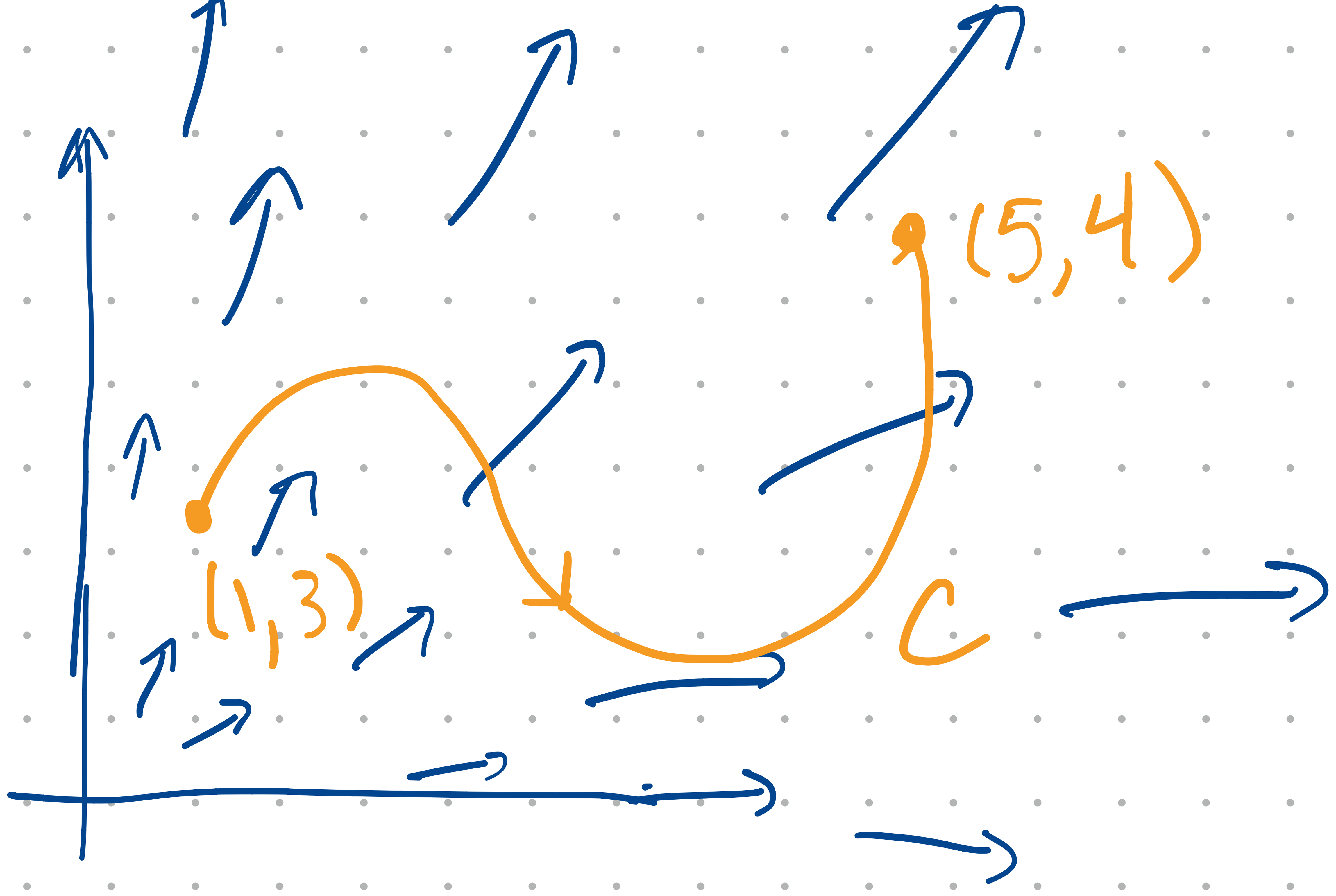
e.g. $f(x,y) = x^2 + y^2$ $\quad\quad$ $\nabla f(x,y) = \langle 2x, 2y \rangle$

⚠ In SVC: every reasonable scalar fn ① is a derivative

But in MVC: not all vector fields ② are gradients.

e.g. $\langle x^2, x^2 + y^2 \rangle$ is not a gradient vector field.

(why?)

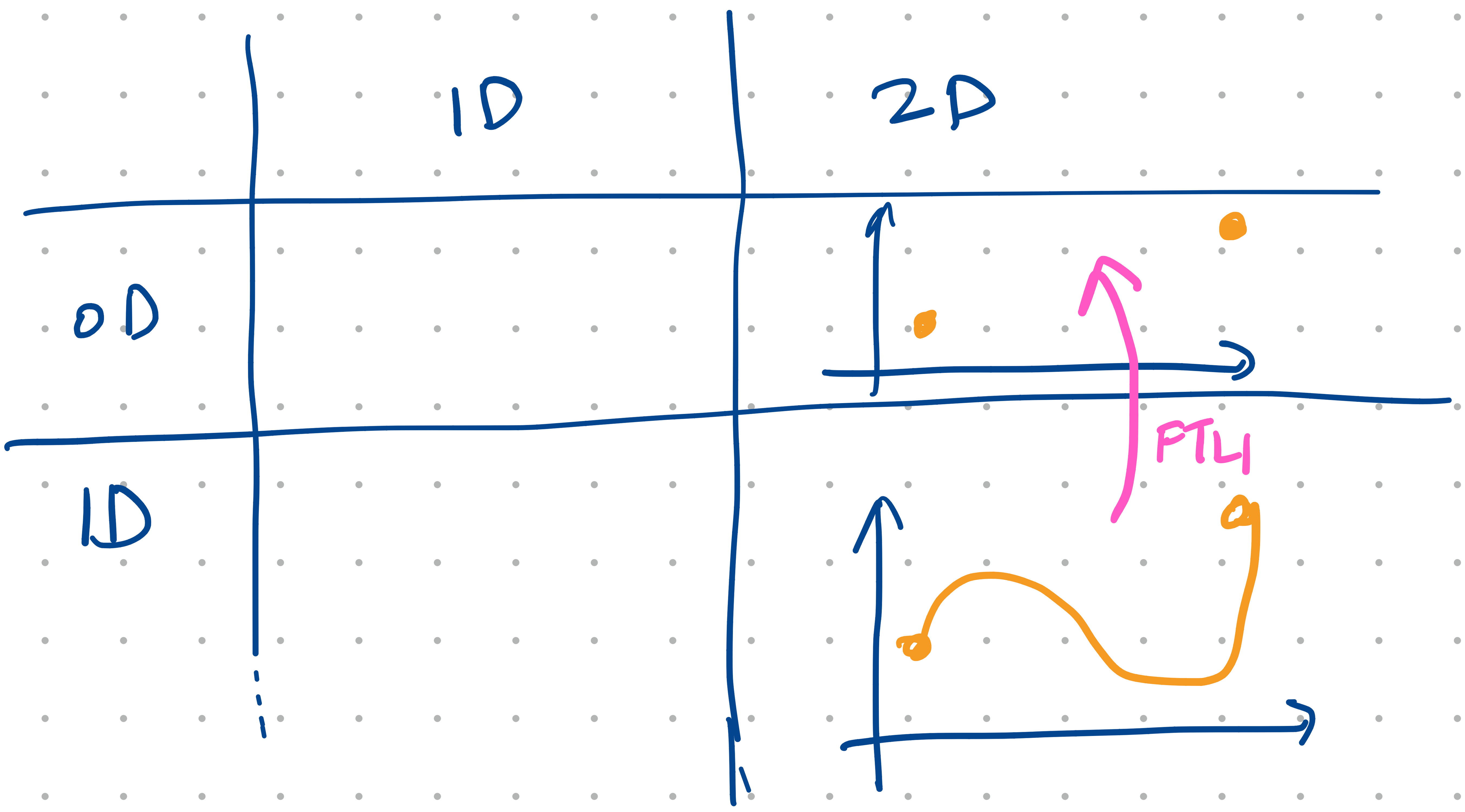


$$\vec{F} = \langle 2x, 2y \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla(x^2+y^2) \cdot d\vec{r}$$

$$= (x^2+y^2)\Big|_{(x,y)=(1,3)}^{(5,4)} = 41 - 10 = 31$$

FTLI



$\vec{F}$ a vec. field defined on region R.

① $\oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed loop in R

$\Longleftrightarrow$

② "path-independence"

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for every pair of paths C_1, C_2 in R that share the same endpoints

$\vec{F}$ is conservative (on R)

① $\Longleftrightarrow$ ③, ② $\Longleftrightarrow$ ③

③ there exists a function f defined on R s.t. ("potential")

$$\nabla f = \vec{F}.$$

③ $\Longrightarrow$ ④ (Clairaut's)

④ $\Longrightarrow$ ③ if R is "simply connected."

④ $\nabla \times \vec{F} = \vec{0}$ (in 2D this means $P_y = Q_x$ where $\vec{F} = \langle P, Q\rangle$)

How to show $\vec{F}$ is conservative?

③ Actually find the potential function.

④ If R is simply connected, then check $P_y = Q_x$.

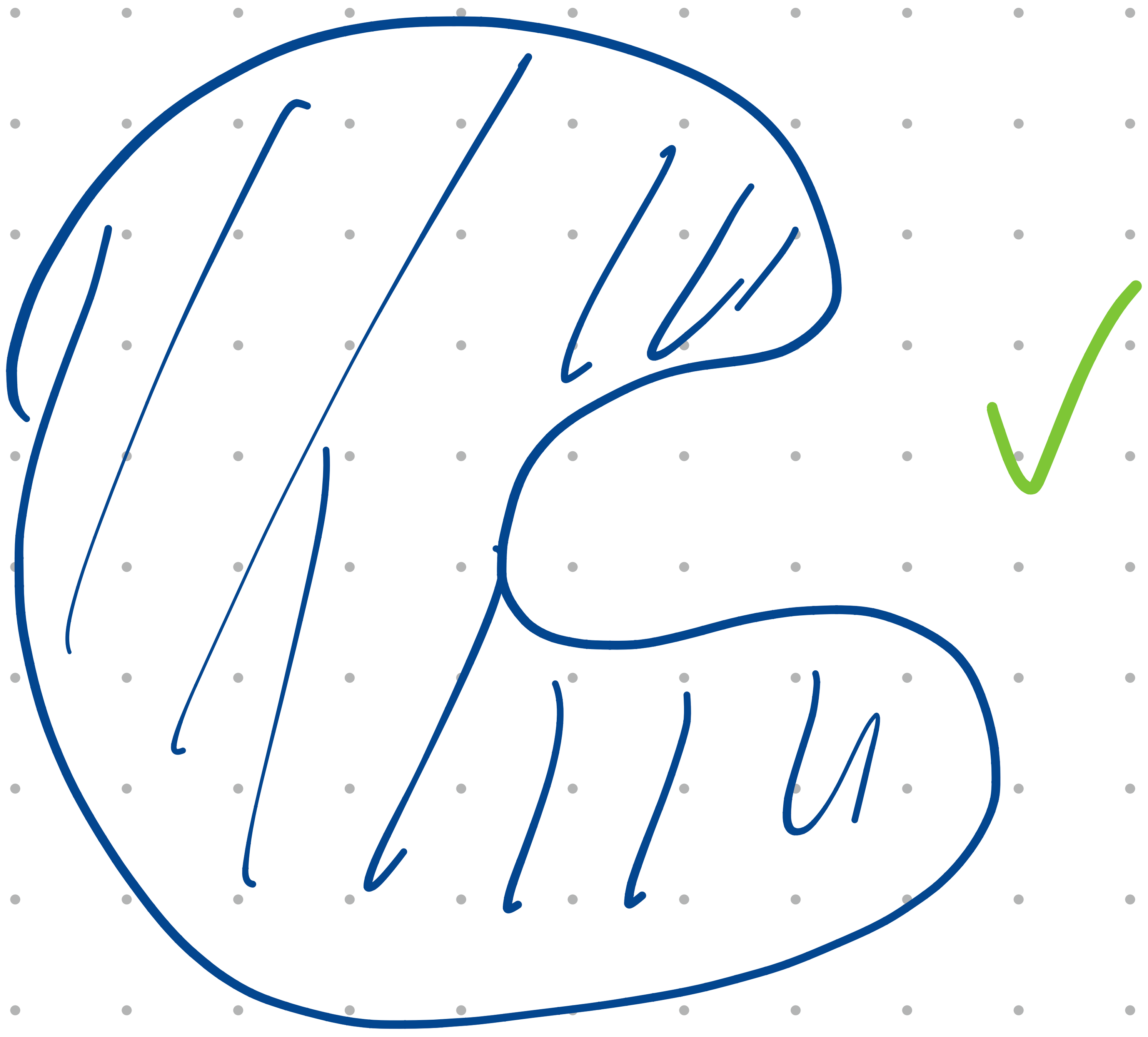
How to show $\vec{F}$ is not conservative?

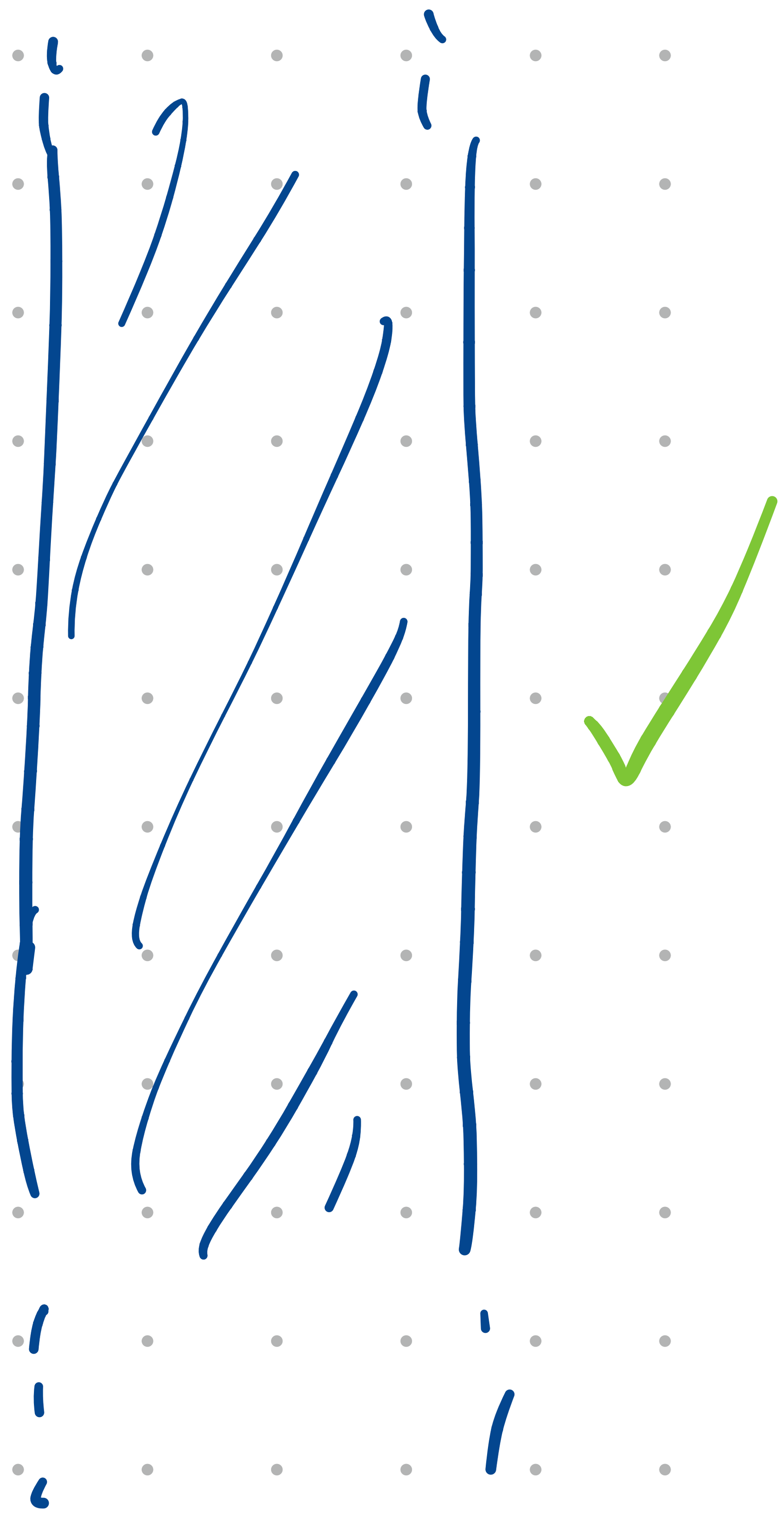
①,② find some loop C s.t. $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

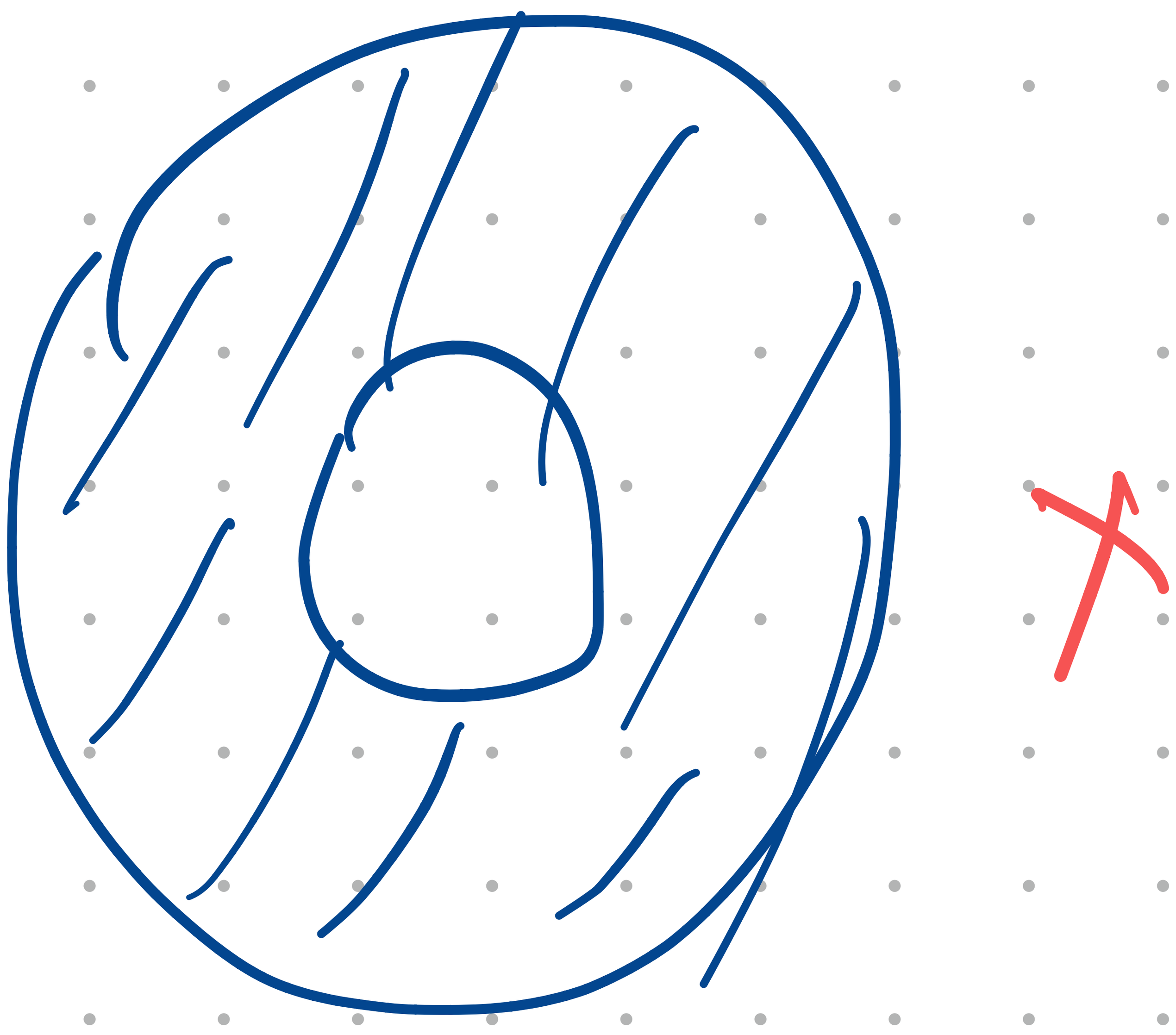
③ Try to integrate and show it results in a contradiction.

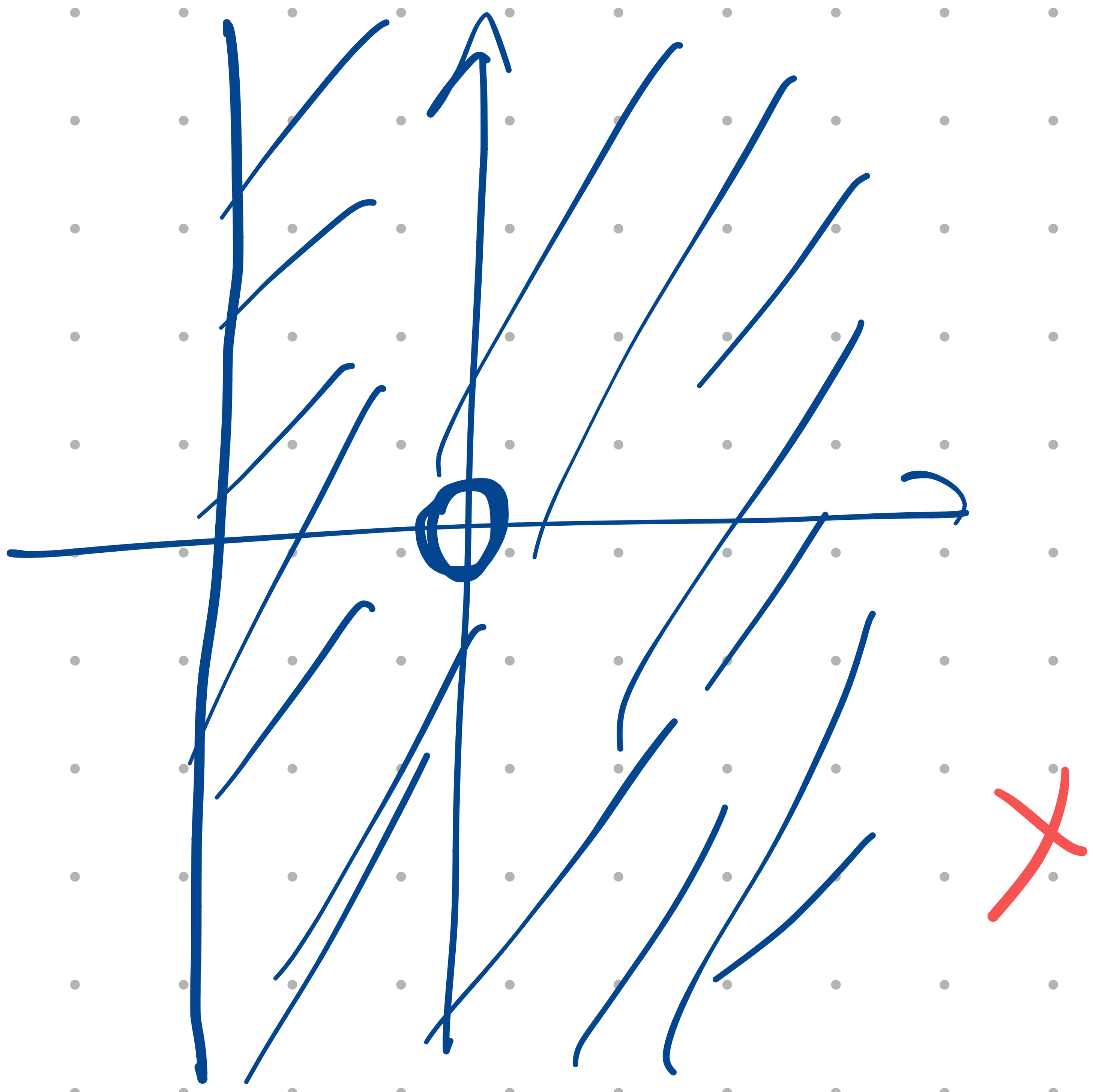
④ show $P_y \neq Q_x$ (recommended)

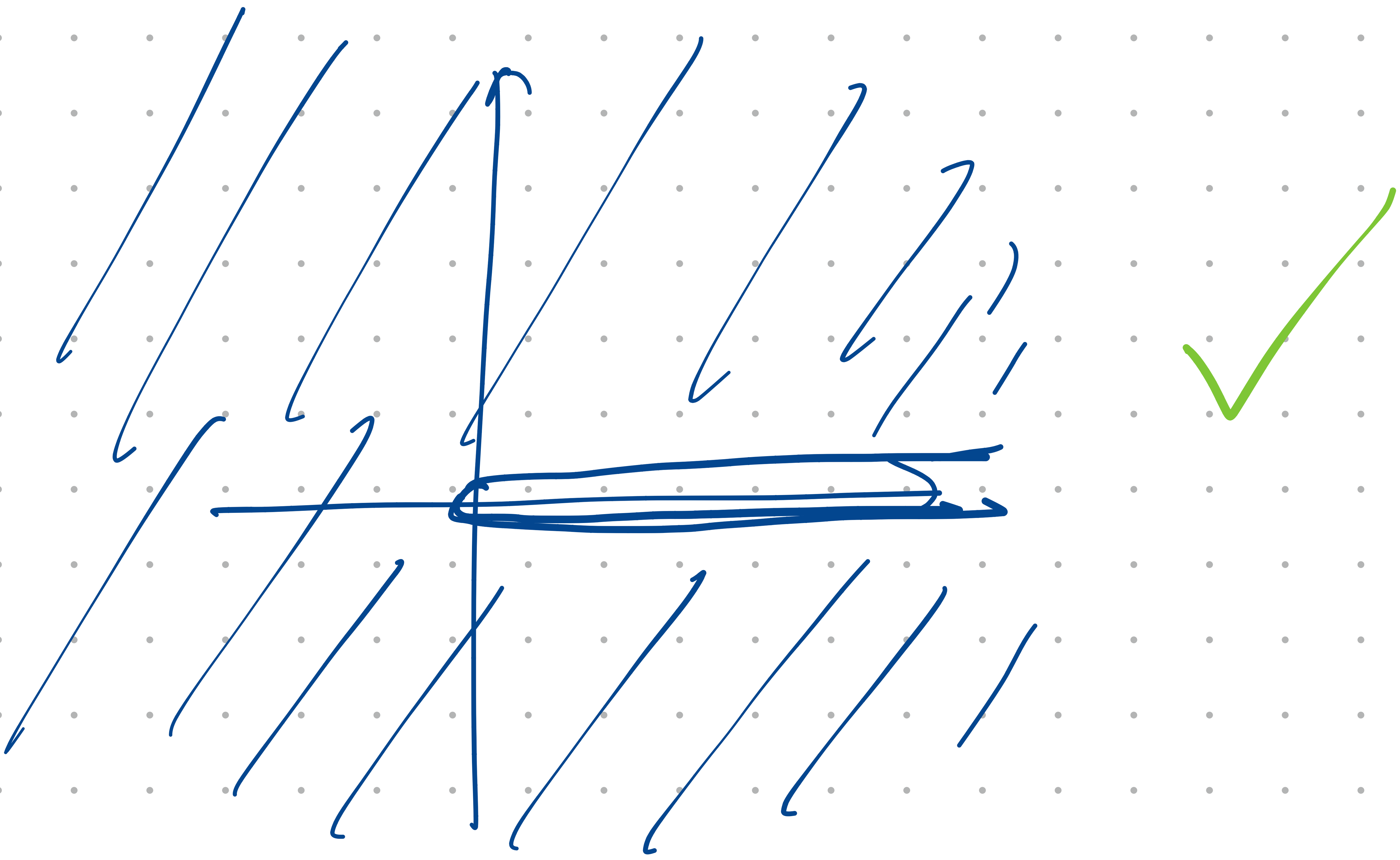
A region in 2D is simply connected if it has no holes. Equivalently, every loop in the region enclosed only points belonging to the region.











1) @ $(0,-5)$

$\leftarrow$ $\rightarrow$ $\downarrow$ $\uparrow$

$\langle x, -y \rangle$ @ $(0,-5)$ is $\langle 0, 5 \rangle$

2) domain is $\mathbb{R}^2$, which is simply conn.

$P_y = 0 = Q_x$ ✓ so it's conservative.

$$f_x(x,y) = x \implies f(x,y) = \frac{1}{2}x^2 + \underbrace{C(y)}_{\text{function of } y}$$

$$\Downarrow$$

$$-y = f_y(x,y) = 0 + C'(y)$$

thus $C(y) = -\frac{1}{2}y^2 + D \leftarrow$ const

so an example of a potential fn is

$$f(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + 42$$

3) $\langle y, -x \rangle$ @ $(0,-5)$ is $\langle -5, 0 \rangle$

i.e. $\longleftarrow$

so first picture.

4) It's not conservative:

$$P_y = 1 \neq -1 = Q_x$$

<u>Alternative:</u>

Assume $\nabla f = \langle y, -x \rangle$.

$$f_x(x,y) = y \Longrightarrow f(x,y) = xy + \underbrace{C(y)}_{\text{fn. of } y}$$

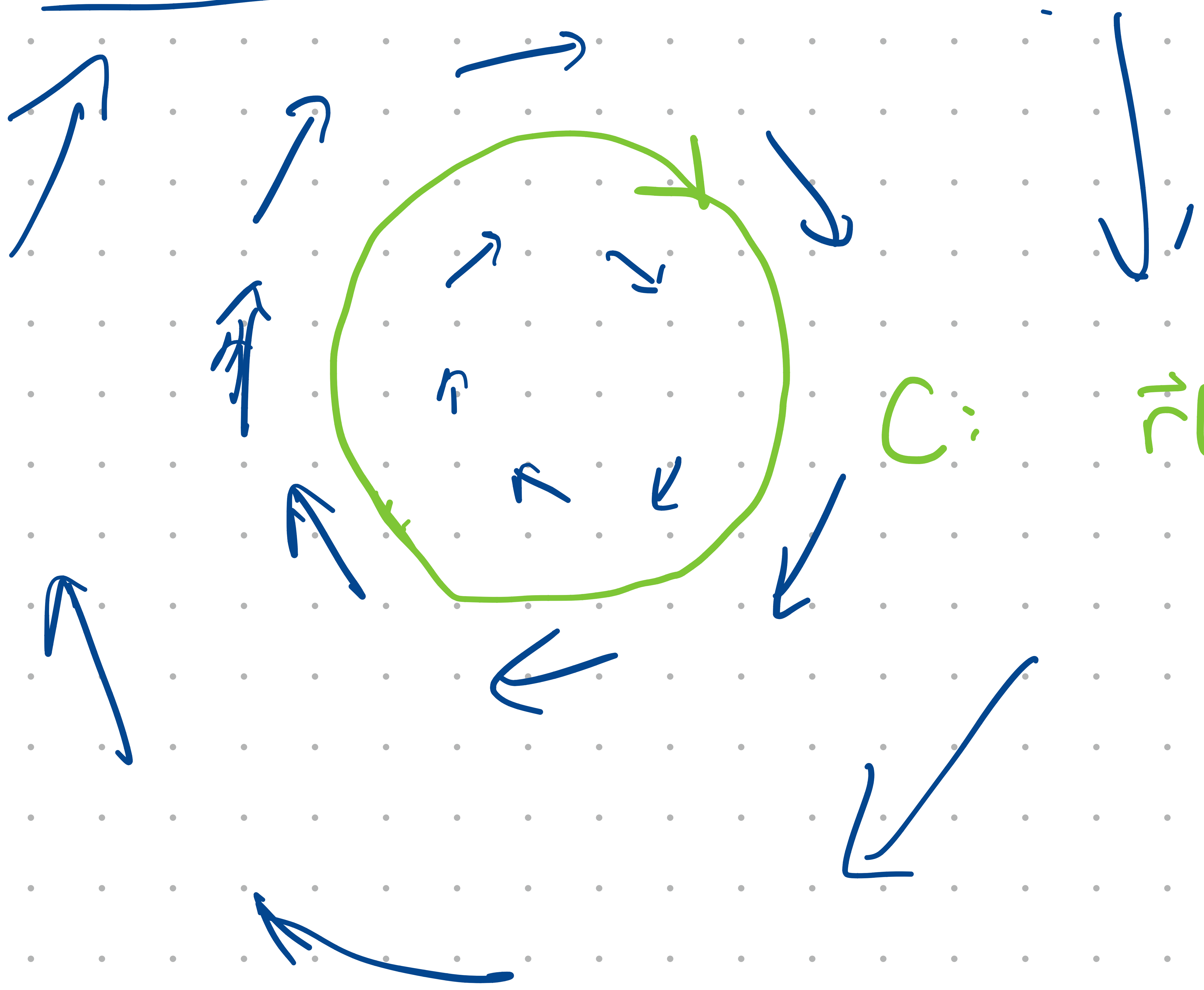
$$\Downarrow$$

$$-x = f_y(x,y) = x + C'(y)$$

$$C'(y) = -2x \quad \text{impossible}$$

involves y only.

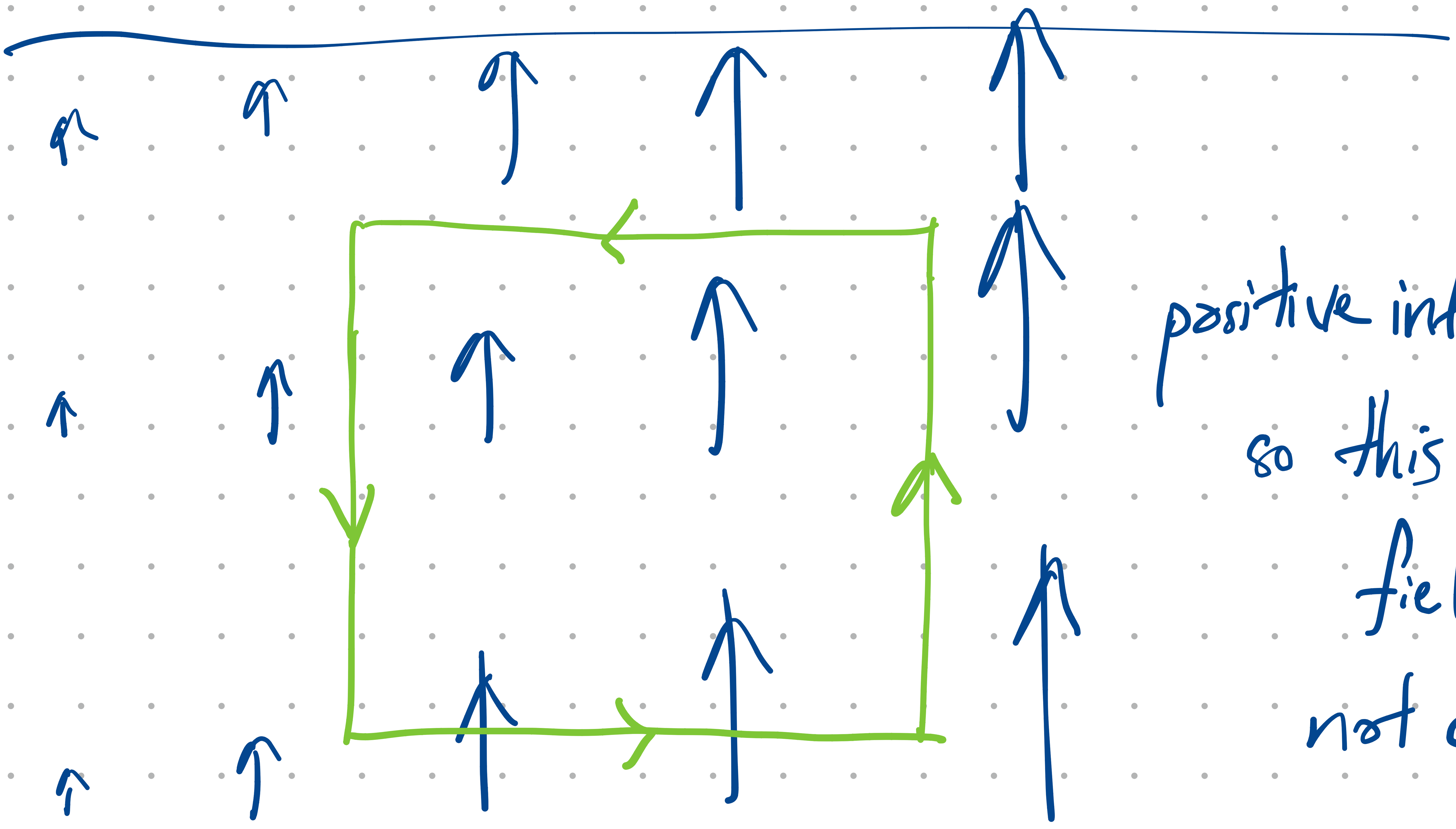
Alternative:



C: $\vec{r}(t) = \langle \sin t, \cos t \rangle$

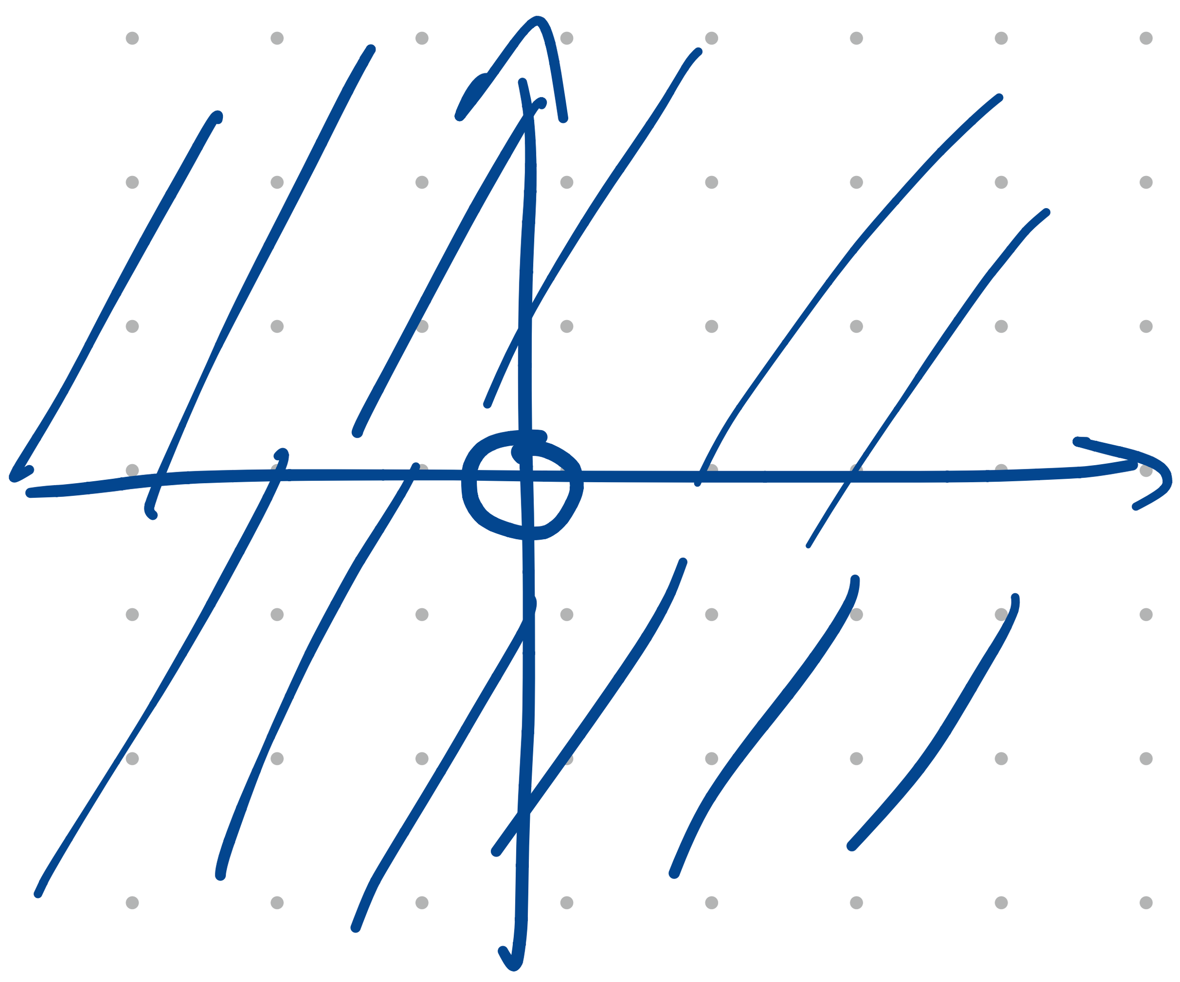
$0 \le t \le 2\pi$

Exercise: $\displaystyle\int_C \langle y, -x \rangle \cdot d\vec{r} = 2\pi \neq 0.$



positive integral so this vec. field is not conservative.

5) $P_y = Q_x$ (check!) however

the domain is  $\mathbb{R}^2 - (0,0)$

and this is not simply connected.

In fact $\vec{F}$ is not conservative.

Exercise: $\displaystyle\oint_C \vec{F} \cdot d\vec{r} \neq 0$

C = a loop around the origin